Understanding SMT Solvers and Their Proofs

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CS Seminar – Union College
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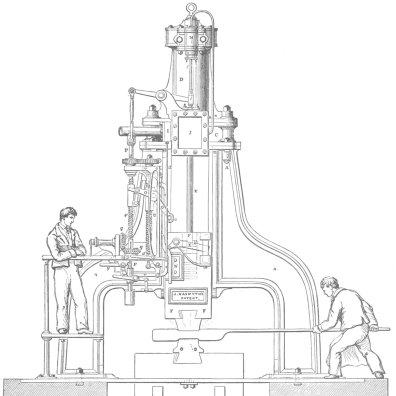

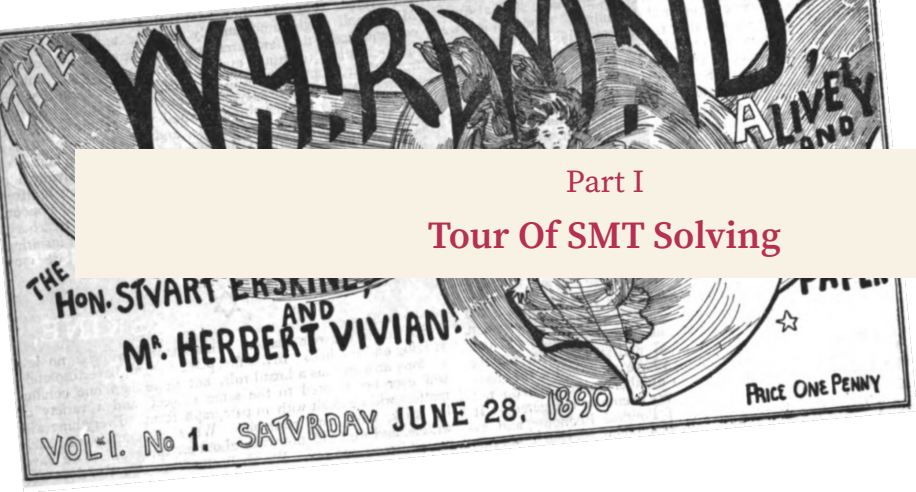

Part I

Tour Of SMT Solving

A Toy Example

1. We produce 1L, 2L, and 3L bottles.
2. The price of a bottle is the volume plus four times the wall thickness (in mm).
3. The price must be less than 4\$.
4. If the new machine is broken, we cannot produce 3L bottles, and the wall thickness must be more than 1mm.
5. The new machine is broken.
6. For all bottle sizes, the all thickness can at most be the volume in liters.

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To solve this, we must understand:

- Logic: **and**, **if then**
- Arithmetic: **four times the wall thickness**
- Universal statements: **for all**

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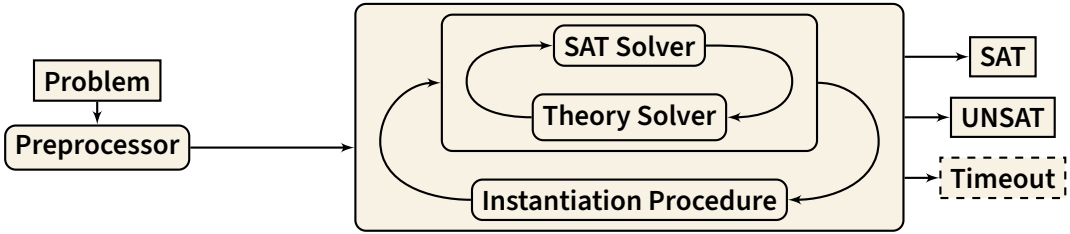
This is **Satisfiability Modulo Theories**

SMT Solving As A Diagram



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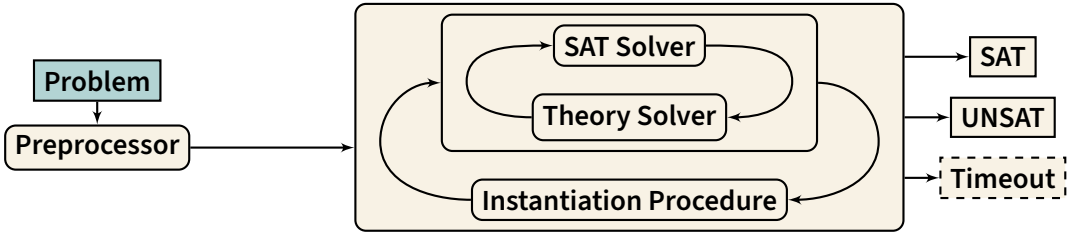

An Example: Problem Specification

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1. $v = 1 \lor v = 2 \lor v = 3$
2. $p = v + 2t$
3. $p < 4$
4. $b \to (v \neq 3 \land t > 1)$
5. b
6. $\forall z.\, v = z \to t \leq z$

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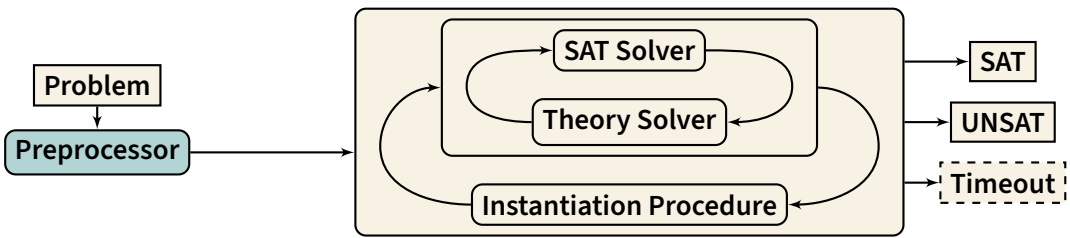

An Example: Preprocessing

1. $v = 1 \vee v = 2 \vee v = 3$
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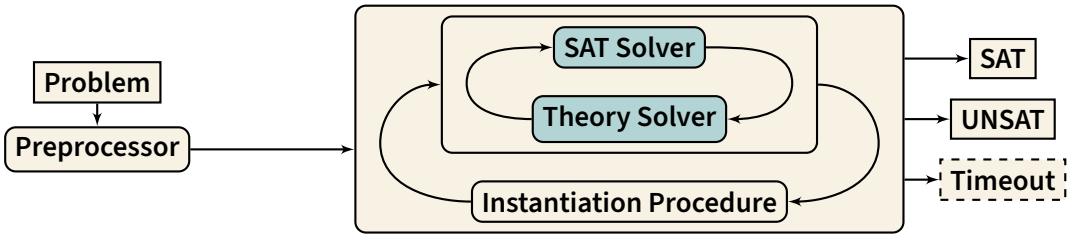

An Example: The Ground Solver

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- $p_1 \vee p_2 \vee p_3$
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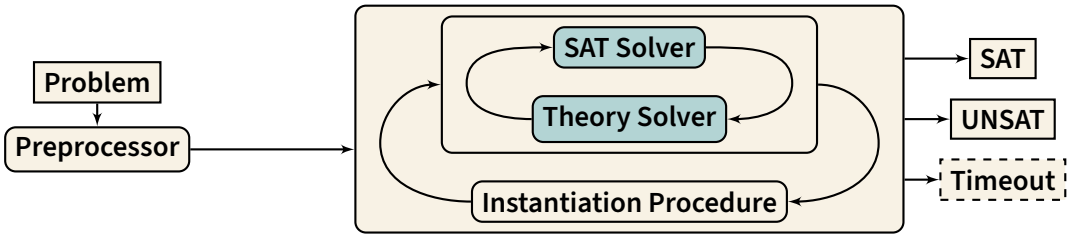

An Example: The SAT Solver and the Theory Solver

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Linear Arithmetic Solver

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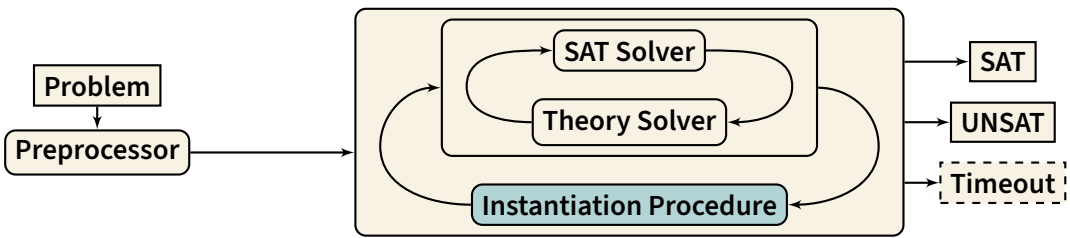
2. That works! 🎉

SMT Solving As A Diagram



An Example: Quantifier Instantiation

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Instantiation Procedure

- I have $\forall z.\, \neg v = z \vee \neg z < t$

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SAT Solver

- That's $\neg p_1 \vee \neg p_5$
- Oh no 😢

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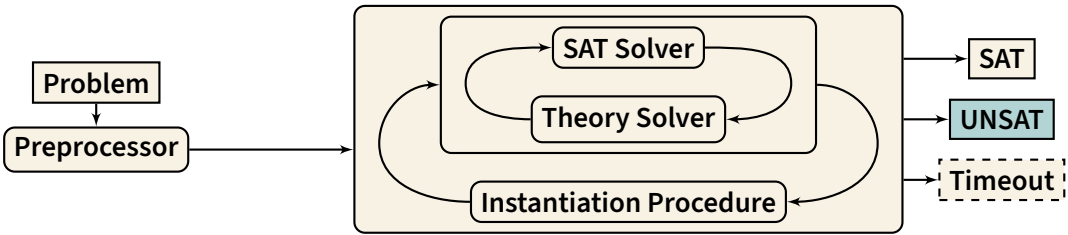
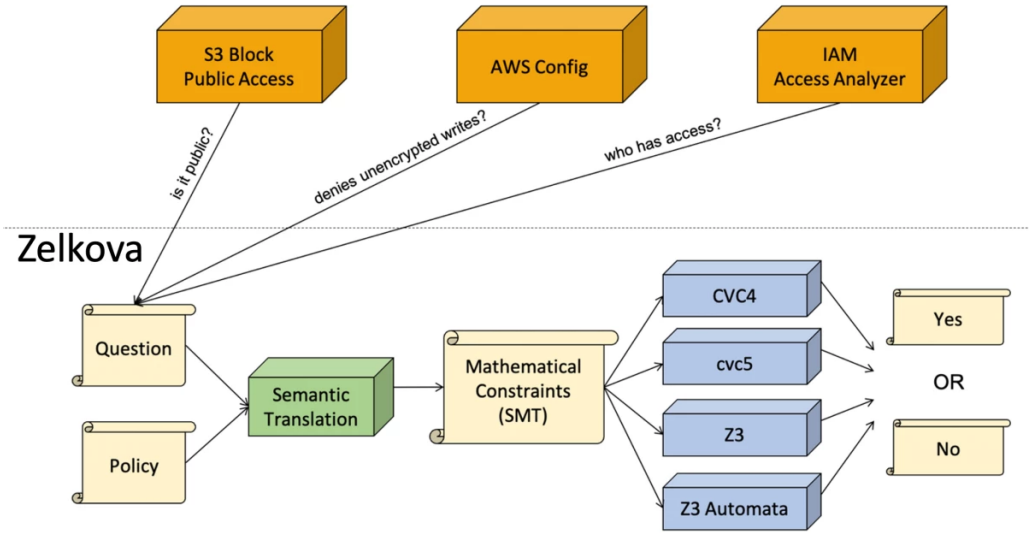


Example Application: aws Zelkova



Using SMT-LIB

```
(set-logic LRA)
(declare-const v Real) (declare-const t Real)
(declare-const b Bool)
(assert (or (= v 1) (= v 2) (= v 3)))
(assert (< (+ v (* 2 t)) p))
(assert (= p 4))
(assert (=> b (and (not (= v 3)) (> t 1))))
(assert b)
(assert (forall ((z Real)) (=> (= v z) (<= t z))))
(check-sat)
```

- 👩‍💼 Most SMT solvers support SMT-LIB
- 🧮 Theories: arithmetic, arrays, data-types, bit-vectors, strings, ...
- 🏅 Yearly competition (SMT-COMP)
- 📚 Large benchmark library

Some Solvers You Can Try (a Biased List)

veriT

- Small solver
- Excellent proofs, good quantifier support
- `www.verit-solver.org`

CVC5

- Industrial strength
- Supports everything
- `cvc5.github.io`

Bitwuzla

- Specialized on bit-vectors, and floating-points
- Very fast
- `bitwuzla.github.io`

Z3

- Very established
- Also supports everything
- `https://github.com/Z3Prover/z3`

Part II

SMT Proofs

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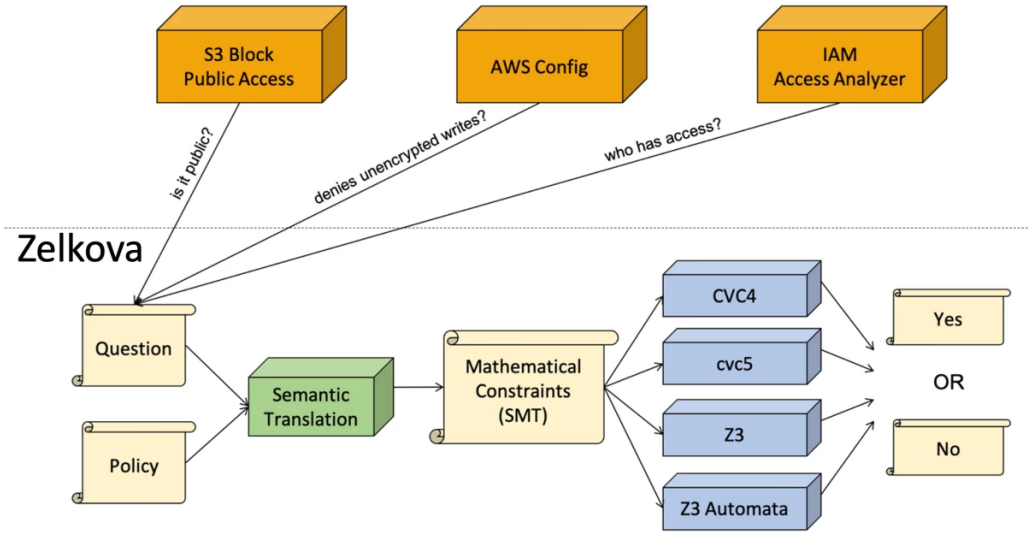

Zelkova Style SMT Constraints

$$Policy \Rightarrow Query \text{ is valid}$$
$$\neg(Policy \Rightarrow Query) \text{ is unsatisfiable}$$
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- Query is against policy: satisfiable!
  - Evidence: counter**model**
  - Easy to check by evaluation.
- Query follows policy: unsatisfiable!
  - Evidence: refutation **proof**
  - Hard!

SMT Proofs: Basic Structure

$$\dfrac{\dfrac{t_2}{t_3}\\ \vdots \\ t_1 \quad t_4}{t_1 \wedge t_4}\ \text{andI}$$

$$t_1, t_2 \vdash t_1 \wedge t_4$$

```
(assume a0 t1)
(assume a1 t2)
(step  s1 t3
       :premises (a1)     :rule rule1)
...
(step s20 t4
       :premises (s19)    :rule rule2)
(step s21 (and t1 t4)
       :premises (a0 s20) :rule andI)
```

SMT Proofs: Alternative View

Proofs as Terms

- Proofs are terms of a dedicated `Proof` type.
- The `Proof` type depends on the formula it proves.

Example

```
(andI
    ((assume t1)
        (rule2 (...(rule1 ((assume t2)))...))
    )
) : Proof (and t1 t4)
```

Eunoia: CVC5's DSL to Describe Proof Rules

and introduction

```
(declare-rule andI ((F1 Bool) (F2 Bool))
    :premises (F1 F2)
    :conclusion (and F1 F2)
)
```

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Resolution

```
(program $resolve ((C1 Bool) (C2 Bool) (pol Bool) (L Bool))
  (Bool Bool Bool Bool) Bool
  (($resolve C1 C2 pol L)
    (eo::list_concat or ($nary_remove or false (eo::ite pol L (not L)) C1)
                        ($nary_remove or false (eo::ite pol (not L) L) C2))
  )
)
(declare-rule resolution ((C1 Bool) (C2 Bool) (pol Bool) (L Bool))
    :premises (C1 C2) :args (pol L) :conclusion ($resolve C1 C2 pol L)
)
```

System Y: A Mechanized Formal Core Language For Eunoia

- Ongoing work!
- How can we know Eunoia is sound?

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- Divergence is handled via guards
  - you must provide evidence a program evaluates in finitely many steps.

Thank You!

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  - Branch variables have the same general type,
  - but type variables instantiation is branch independent.
- Programs can diverge.
- If there is no matching branch, they get stuck!